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On the Jaynes–Cummings model with multiphoton transitions in a cavity

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Abstract. The Jaynes-Cummings model with multiphoton transitions in a cavity is examined, and an exact solution of the master equation for the density matrix is found. Absorption and emission spectra are investigated.

1. Introduction

The Jaynes-Cummings model (Jaynes and Cummings 1963, Yoo and Eberly 1985) of a two-level atom interacting with the electromagnetic field in an ideal cavity is one of the few exactly soluble models in quantum optics. It enables one to calculate all the quantum-mechanical properties of a system. It predicts many interesting effects such as vacuum field Rabi oscillations (Sanchez-Mondragon *et al* 1983, Agarwal 1984, 1985), quantum collapse and revival (Yoo and Eberly 1985 and references therein). The quantum collapse and revival have been observed experimentally by Rempe *et al* (1987).

In recent papers Agarwal and Puri (1986a, b), Barnett and Knight (1986) and Filipowicz *et al* (1986) have studied the effects of dissipation in the Jaynes-Cummings model and their influence on revivals and other quantum features; in particular, the absorption and emission spectra have been calculated. Single-mode m-photon absorption and m-photon emission processes in a two-level atomic system have been considered by Zubairy and Yeh (1980). Other multiphoton processes in a lossless cavity have recently been extensively investigated in a number of papers (Singh 1982, Mavroyannis 1985, Allen and Stroud 1982, Eberly and Krasinski 1984, Shumovsky *et al* 1985, 1986).

In this paper we consider the Jaynes-Cummings model with multiphoton transitions in the presence of cavity-relaxation effects. In order to solve the problem we follow the procedure presented by Agarwal and Puri (1986) and Shumovsky *et al* (1985, 1986).

2. Solution for density–matrix elements

The Jaynes-Cummings model with multiphoton transitions describes the interaction of a single-model electromagnetic field with a two-level atom via m-photon processes.

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The Hamiltonian for this model in the RWA and electric dipole approximation is

$$H = \hbar\omega_0 S^z + \hbar\omega a^+ a + \hbar g(a^{+m} S^- + a^m S^+) \tag{1}$$

where $S^{\pm,z}$ are the spin-$\frac{1}{2}$ operators, a (a^+) is the annihilation (creation) operator of the radiation field. The parameter g is the constant of atom-mode coupling. Here ω_0 is the transition frequency of the atom and ω is the mode frequency, and they obey the condition:

$$\omega_0 - m\omega = \Delta \tag{2}$$

where Δ is the detuning parameter. In Hamiltonian (1) the Stark shifts of the atomic levels have been ignored (Agarwal 1985).

Further, we shall assume that a field can decay at the rate 2κ. The density matrix for the combined atom–field system by the standard master equation techniques is (Agarwal 1986a, b, Barnett and Knight 1986)

$$\partial\rho/\partial t = \mathrm{i}[H, \rho] - \kappa(a^+ a\rho - 2a\rho a^+ + \rho a^+ a) = L\rho. \tag{3}$$

The Hamiltonian H causes transitions between the states $|n, e\rangle$ and $|n+m, g\rangle$. Field and atom occupation numbers change at the same time. The relaxation in the cavity only changes the photon number. For example, if the initial state of the system is $|n, g\rangle$, then the system can be found in any of the states

$$|p, g\rangle \qquad p = 0, 1, \ldots, n$$

$$|q, e\rangle \qquad q = 0, 1, \ldots, n-m.$$

For the initial state $|0, e\rangle$, the states to be considered are $|m, g\rangle, |m-1, g\rangle, \ldots, |0, g\rangle$. The density matrix elements now satisfy

$$\langle 0, g|\dot\rho|0, e\rangle = \mathrm{i}(m\omega + \Delta)\langle 0, g|\rho|0, e\rangle + \mathrm{i}g\sqrt{m!}\langle 0, g|\rho|m, g\rangle \tag{4}$$

$$\langle 0, g|\dot\rho|m, g\rangle = (\mathrm{i}m\omega - \kappa m)\langle 0, g|\rho|m, g\rangle + \mathrm{i}g\sqrt{m!}\langle 0, g|\rho|0, e\rangle. \tag{5}$$

The results following from (4) and (5) are

$$\begin{aligned}\langle 0, g|\rho|0, e\rangle = \frac{1}{z_1 - z_2}\{&[(z_1 - \mathrm{i}m\omega + \kappa m)\langle 0, g|\rho(0)|0, e\rangle \\ &+ \mathrm{i}g\sqrt{m!}\langle 0, g|\rho(0)|m, g\rangle]\,\mathrm{e}^{z_1 t} - [(z_2 - \mathrm{i}m\omega + \kappa m) \\ &\times\langle 0, g|\rho(0)|0, e\rangle + \mathrm{i}g\sqrt{m!}\langle 0, g|\rho(0)|m, g\rangle]\,\mathrm{e}^{z_2 t}\}\end{aligned} \tag{6}$$

$$\begin{aligned}\langle 0, g|\rho|m, g\rangle = \frac{1}{z_1 - z_2}\{&[(z_1 - \mathrm{i}m\omega - \mathrm{i}\Delta)\langle 0, g|\rho(0)|m, g\rangle \\ &+ \mathrm{i}g\sqrt{m!}\langle 0, g|\rho(0)|0, e\rangle]\mathrm{e}^{z_1 t} - [(z_2 - \mathrm{i}m\omega - \mathrm{i}\Delta) \\ &\times\langle 0, g|\rho(0)|m, g\rangle + \mathrm{i}g\sqrt{m!}\langle 0, g|\rho(0)|0, e\rangle]\,\mathrm{e}^{z_2 t}\}\end{aligned} \tag{7}$$

$$z_{1,2} = \mathrm{i}(m\omega + \tfrac{1}{2}\Delta) - \tfrac{1}{2}\kappa m \pm \tfrac{1}{2}[(\kappa m + \mathrm{i}\Delta)^2 - 4g^2 m!]^{1/2}. \tag{8}$$

Using these solutions we can calculate an absorption spectrum for the model, assuming in addition that our model interacts with a weak-probe field. For Rydberg atoms the probe field will be a microwave field of frequency ν. Considering the selection rule we assume that the absorption of the probe field is one- or two-photon processes if m is odd or even, respectively. Then, the master equation (3) will be

$$\partial\rho/\partial t = L\rho - \mathrm{i}[(G^{(\varepsilon)} S^+ \mathrm{e}^{\mathrm{i}\varepsilon\nu t} + \mathrm{HC}), \rho] \tag{9}$$

where $\varepsilon=1$ if m is odd and $\varepsilon=2$ if m is even. The values $G^{(1)}$ and $G^{(2)}$ are the coupling constants of the atom with the microwave field via one-photon (if m is odd) or two-photon (if m is even) processes.

Using the perturbation procedure (Agarwal and Puri 1986a) one can calculate the time-average rate of absorption W in the form

$$W=-2\varepsilon\nu|G^{(\varepsilon)}|^2 \operatorname{Re}\int_0^\infty \mathrm{d}\tau\, \mathrm{e}^{-\mathrm{i}\varepsilon\nu\tau}\operatorname{Tr}(S^+ \mathrm{e}^{L\tau}[S^-,\rho(0)]). \tag{10}$$

In the case, when our cavity is at zero temperature, the initial density matrix $\rho(0)$ is

$$\rho(0)=|0,g\rangle\langle 0,g| \tag{11}$$

and hence

$$W=2\varepsilon\nu|G^{(\varepsilon)}|^2 \operatorname{Re}\int_0^\infty \mathrm{d}\tau\, \mathrm{e}^{-\mathrm{i}\varepsilon\nu\tau}\operatorname{Tr}[S^+ \mathrm{e}^{L\tau}|0,g\rangle\langle 0,e|]. \tag{12}$$

The operator $\mathrm{e}^{L\tau}|0,g\rangle\langle 0,e|$ satisfies (4) and hence

$$\mathrm{e}^{L\tau}|0,g\rangle\langle 0,e|=\alpha(\tau)|0,g\rangle\langle 0,e|+\beta(\tau)|0,g\rangle\langle m,g| \tag{13}$$

where

$$\alpha(\tau)=\frac{z_1-\mathrm{i}m\omega+\kappa m}{z_1-z_2}\mathrm{e}^{z_1\tau}-\frac{z_2-\mathrm{i}m\omega+\kappa m}{z_1-z_2}\mathrm{e}^{z_2\tau} \tag{14}$$

$$\beta(\tau)=\frac{\mathrm{i}g\sqrt{m!}}{z_1-z_2}\mathrm{e}^{z_1\tau}-\frac{\mathrm{i}g\sqrt{m!}}{z_1-z_2}\mathrm{e}^{z_2\tau}. \tag{15}$$

Substituting (13) in (12) and simplifying (12) we get

$$W=2\varepsilon\nu|G^{(\varepsilon)}|^2\operatorname{Re}\hat{\alpha}(\mathrm{i}\varepsilon\nu) \tag{16}$$

where

$$\hat{\alpha}(\mathrm{i}\varepsilon\nu)=\int_0^\infty \mathrm{e}^{\mathrm{i}\varepsilon\nu\tau}\alpha(\tau)\,\mathrm{d}\tau.$$

We will consider the case of exact resonance when $\Delta=\omega_0-m\omega=0$ and

$$z_{1,2}=\mathrm{i}\omega_0-\tfrac{1}{2}\kappa m\pm\tfrac{1}{2}(\kappa^2m^2-4g^2m!)^{1/2}. \tag{17}$$

Here, we can consider the following cases:

(i) $$\kappa^2m^2-4g^2m!>0. \tag{18}$$

In this case, from (14), (16) and (17) we get

$$W=2\varepsilon\nu|G^{(\varepsilon)}|^2\frac{g^2m!}{(\kappa^2m^2-4g^2m!)^{1/2}}\left\{\frac{1}{(\varepsilon\nu-\omega_0)^2+\frac{1}{4}[\kappa m-(\kappa^2m^2-4g^2m!)^{1/2}]^2}\right.$$
$$\left.+\frac{1}{(\varepsilon\nu-\omega_0)^2+\frac{1}{4}[\kappa m+(\kappa^2m^2-4g^2m!)^{1/2}]^2}\right\}. \tag{19}$$

In the case of a bad cavity $\kappa^2m^2\gg 4g^2m!$ the spectrum (19) has only a single peak in the position $\varepsilon\nu=\omega_0$.

(ii) $$\kappa^2m^2-4g^2m!<0. \tag{20}$$

In this case by substituting (14) and (17) into (16), we obtain

$$W = 2\varepsilon\nu|G^{(\varepsilon)}|^2 \frac{\frac{1}{2}\kappa m}{(4g^2m! - \kappa^2 m^2)^{1/2}} \times \left\{ \frac{\varepsilon\nu - \omega_0 + (4g^2m! - \kappa^2m^2)^{1/2}}{[\varepsilon\nu - \omega_0 + \frac{1}{2}(4g^2m! - \kappa^2m^2)^{1/2}]^2 + \frac{1}{4}\kappa^2m^2} - \frac{\varepsilon\nu - \omega_0 - (4g^2m! - \kappa^2m^2)^{1/2}}{[\varepsilon\nu - \omega_0 - \frac{1}{2}(4g^2m! - \kappa^2m^2)^2 + \frac{1}{4}\kappa^2m^2} \right\}. \tag{21}$$

For the good cavity case, $\kappa^2m^2 \ll 4g^2m!$, (21) shows that the spectrum is a doublet $\varepsilon\nu = \omega_0 \pm g\sqrt{m!}$, the width of each doublet being $\kappa m/2$. It should be noted that in the case of multiphoton absorption the widths of lines are proportional to m.

3. Emission spectra for multiphoton processes

Following Agarwal and Puri (1986a), we define the transient spectrum of the radiation that leaks out as

$$S(\nu, T) = 2\Gamma\beta \,\mathrm{Re} \sum_{ij} A_{ij}(2\Gamma + \xi_i)^{-1} [\![(\Gamma + \xi_j + i\nu - \lambda_i)^{-1} \times \{\exp(-\xi_j T) - \exp[-T(\Gamma - \lambda_i + i\nu)]\} - (\Gamma + \lambda_i - i\nu)^{-1}\{\exp[-(\Gamma + i\nu - \lambda_i)T] - \exp(-2\Gamma T)\}]\!], \tag{22}$$

where we assume that the correlation function has the structure

$$\langle a^+(t+\tau)a(t)\rangle = \sum_{ij} A_{ij} \exp(\lambda_i\tau + \xi_j t). \tag{23}$$

Γ is the bandwidth of the detector, T is the time at which the spectrum is evaluated and β is a measure of the leakage of the field energy.

Using regression theory, one can show that

$$\langle a^+(t+\tau)a(t)\rangle = \mathrm{Tr}[a^+ \,\mathrm{e}^{L\tau} a \,\mathrm{e}^{Lt}\rho(0)] \tag{24}$$

where the initial density matrix $\rho(0)$ is $|0, e\rangle\langle 0, e|$. This initial state is chosen keeping in view the problem of pure spontaneous emission.

Using (3) we define equations of motion for the operator

$$\begin{aligned} &\mathrm{e}^{Lt}|0, e\rangle\langle 0, e| \equiv (|0, e\rangle\langle 0, e|)_t \\ &(d/dt)(|0, e\rangle\langle 0, e|)_t = -\mathrm{i}g\sqrt{m!}(|m, g\rangle\langle 0, e|)_t + \mathrm{i}g\sqrt{m!}(|0, e\rangle\langle m, g|)_t. \end{aligned} \tag{25}$$

Resulting is a closed set of equations

$$\left[\frac{\mathrm{d}}{\mathrm{d}t} + \begin{bmatrix} 0 & \mathrm{i}g\sqrt{m!} & -\mathrm{i}g\sqrt{m!} & 0 \\ \mathrm{i}g\sqrt{m!} & (\kappa m - \mathrm{i}\Delta) & 0 & -\mathrm{i}g\sqrt{m!} \\ -\mathrm{i}g\sqrt{m!} & 0 & (\kappa m + \mathrm{i}\Delta) & \mathrm{i}g\sqrt{m!} \\ 0 & -\mathrm{i}g\sqrt{m!} & \mathrm{i}g\sqrt{m!} & 2\kappa m \end{bmatrix}\right] \begin{bmatrix} |0, e\rangle\langle 0, e| \\ |m, g\rangle\langle 0, e| \\ |0, e\rangle\langle m, g| \\ |m, g\rangle\langle m, g| \end{bmatrix} = 0. \tag{26}$$

These equations are solved by the Laplace transforms, the results of which are given by the matrix relation

$$\hat{\psi}(z) = P^{-1}(z)$$
$$\times \begin{vmatrix} z_0(z_0^2-\kappa^2m^2+4g^2m!) & \mathrm{i}\Delta(z_0^2-\kappa^2m^2) & 2\Delta\kappa g\sqrt{m!} & 2\Delta z_0 g\sqrt{m!} \\ \mathrm{i}\Delta(z_0^2-\kappa^2m^2) & z_0(z_0^2-\kappa^2m^2) & 2\mathrm{i}z_0\kappa mg\sqrt{m!} & -2\mathrm{i}z_0 g\sqrt{m!} \\ 2\Delta\kappa mg\sqrt{m!} & -2\mathrm{i}z_0\kappa mg\sqrt{m!} & z_0(z_0^2+\Delta^2+4g^2m!) & \kappa m(z_0^2+\Delta^2) \\ 2\Delta z_0 g\sqrt{m!} & -2\mathrm{i}z_0^2 g\sqrt{m!} & \kappa m(z_0^2+\Delta^2) & z_0(z_0^2+\Delta^2) \end{vmatrix} \psi(0) \tag{27}$$

$$z_0 = z + m\kappa \qquad \psi_{1,2} = \langle m, g|\rho|0, e\rangle \pm \langle 0, e|\rho|m, g\rangle$$
$$\psi_{3,4} = \langle 0, e|\rho|0, e\rangle \pm \langle m, g|\rho|m, g\rangle$$

where the polynomial $P(z)$ is

$$P(z) = (z+m\kappa)^4 + (z+m\kappa)^2(\Delta^2+4g^2m!-\kappa^2m^2) - \kappa^2m^2\Delta^2. \tag{28}$$

If we denote by M the 4×4 square matrix in (26), then it can be shown that

$$\mathrm{e}^{L\tau}a\,\mathrm{e}^{Lt}\rho(0) = (\mathrm{e}^{-Mt})_{12}\,\mathrm{e}^{L\tau}\sqrt{m}|m-1, g\rangle\langle 0, e| + (\mathrm{e}^{-Mt})_{14}\,\mathrm{e}^{L\tau}\sqrt{m}|m-1, g\rangle\langle m, g|. \tag{29}$$

We can further show that for the operators

$$\mathrm{e}^{L\tau}|m-1, g\rangle\langle 0, e| = (|m-1, g\rangle\langle 0, e|)_\tau$$

and

$$\mathrm{e}^{L\tau}|m-1, g\rangle\langle m, g| = (|m-1, g\rangle\langle m, g|)_\tau$$

we have a closed set of equations

$$\left[\frac{\mathrm{d}}{\mathrm{d}t} + \begin{bmatrix} -\mathrm{i}(\omega+\Delta)+(m-1)\kappa & -\mathrm{i}g\sqrt{m!} \\ -\mathrm{i}g\sqrt{m!} & (2m-1)\kappa-\mathrm{i}\omega \end{bmatrix}\right]\begin{bmatrix} |m-1, g\rangle\langle 0, e| \\ |m-1, g\rangle\langle m, g| \end{bmatrix} = 0. \tag{30}$$

The following results are obtained from (30):

$$\begin{aligned}(|m-1, g\rangle\langle 0, e|)_\tau = \frac{1}{x_1-x_2}\{&[(x_1-\mathrm{i}\omega+(2m-1)\kappa)|m-1, g\rangle\langle 0, e| \\ &+\mathrm{i}g\sqrt{m!}|m-1, g\rangle\langle m, g|]\,\mathrm{e}^{x_1\tau} - [(x_2-\mathrm{i}\omega+(2m-1)\kappa) \\ &\times|m-1, g\rangle\langle 0, e| + \mathrm{i}g\sqrt{m!}|m-1, g\rangle\langle m, g|]\,e^{x_2\tau}\}.\end{aligned} \tag{31}$$

$$\begin{aligned}(|m-1, g\rangle\langle m, g|)_\tau = \frac{1}{x_1-x_2}\{&[(x_1-\mathrm{i}(\omega+\Delta)+(m-1)\kappa) \\ &\times|m-1, g\rangle\langle m, g| + \mathrm{i}g\sqrt{m!}|m-1, g\rangle\langle 0, e|]\,\mathrm{e}^{x_1\tau} \\ &-[(x_2-\mathrm{i}(\omega+\Delta)+(m-1)\kappa)|m-1, g\rangle\langle m, g| \\ &+\mathrm{i}g\sqrt{m!}|m-1, g\rangle\langle 0, e|]\,\mathrm{e}^{x_2\tau}\}\end{aligned} \tag{32}$$

$$x_{1,2} = \mathrm{i}\left(\omega+\frac{\Delta}{2}\right) + \kappa\left(1-\frac{3m}{2}\right) \pm \tfrac{1}{2}[(\kappa m+\mathrm{i}\Delta)^2 - 4g^2m!]^{1/2}. \tag{33}$$

Let us denote the 2×2 square matrix in (30) by N. Then, using the solution of (30) in (29), one can show that

$$\langle a^+(t+\tau)a(t)\rangle = m(\mathrm{e}^{-Mt})_{12}(\mathrm{e}^{-N\tau})_{12} + m(\mathrm{e}^{-Mt})_{14}(\mathrm{e}^{-N\tau})_{22}. \tag{34}$$

The relevant elements of $e^{-N\tau}$ are given by (31) and (32)

$$(e^{-N\tau})_{12}=\frac{ig\sqrt{m!}}{x_1-x_2}(e^{x_1\tau}-e^{x_2\tau})$$

$$(e^{-N\tau})_{22}=\frac{1}{x_1-x_2}[(x_1-i(\omega+\Delta)+(m-1)\kappa)\,e^{x_1\tau}-(x_2-i(\omega+\Delta)+(m-1)\kappa)\,e^{x_2\tau}]. \tag{35}$$

A complete spectrum of spontaneous emission can now be obtained using (34) and (23) in (22). In the long-time limit $\Gamma T\gg 1$ the spontaneous emission spectra consist of several lines whose positions and widths are determined by

$$\mathrm{Im}(\lambda_i-\eta_j) \qquad \Gamma+\mathrm{Re}(\eta_j-\lambda_i).$$

For the case of a good cavity on resonance, (33) shows that the emission spectrum has the form of a doublet, the lines of which are positioned at $\nu=\omega\pm g\sqrt{m!}$ and have width $\Gamma+\kappa(\frac{3}{2}m-1)$. On the other hand, for large Δ spontaneous emission lines occur at the positions $\omega+\Delta$, ω and their widths are $\Gamma+\kappa(m-1)$ and $\Gamma+\kappa(2m-1)$ respectively.

It should be noted in the case $m=1$ our results reduce to those obtained by Agarwal and Puri (1986a).

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References

Agarwal G S 1984 *Phys. Rev. Lett.* **53** 1732
—— 1985 *J. Opt. Soc. Am.* B **2** 480
Agarwal G S and Puri R R 1986a *Phys. Rev.* A **33** 1757
—— 1986b *Phys. Rev.* A **33** 3610
Allen L, Strond C R Jr 1982 *Phys. Rep.* **91** 1-29
Barnett S M and Knight P L 1986 *Phys. Rev.* **133** 2444
Eberly J H and Krasinski J 1984 *Advances in Multiphoton Processes and Spectroscopy* ed S H Lin (Singapore: World Scientific)
Filipowicz P, Javanainen J and Meystre P 1986 *Phys. Rev.* A **34** 3077
Jaynes E T and Cummings F W 1963 *Proc. IEEE* **51** 89
Mavroyannis C 1985 *J. Chem. Phys.* **82** 3563
Rempe G, Walthel H and Klein N 1987 *Phys. Rev. Lett.* **58** 353
Sanchez-Mondragon J J, Narozhny N B and Eberly J H 1983 *Phys. Rev. Lett.* **51** 550
Shumovsky A S, Aliskenderov E I and Fam Le Kien 1985 *J. Phys. A: Math. Gen.* **18** L1031
Shumovsky A S, Aliskenderov E I, Fam Le Kien and Nguyen Dinh Vinh 1986 *J. Phys. A: Math. Gen.* **18** 3607
Singh S 1982 *Phys. Rev.* A **25** 3206
Yoo H I and Eberly J H 1985 *Phys. Rep.* **118** 1-239
Zubairy M S and Yeh J J 1980 *Phys. Rev.* A **21** 1624